SEMIGLOBAL RESULTS FOR $\overline{\partial}$ ON A COMPLEX SPACE WITH ARBITRARY SINGULARITIES

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Abstract. We obtain some L^2-results for the $\overline{\partial}$ operator on forms that vanish to high order near the singular set of a complex space.

1. Introduction

Let X be a pure $n-$dimensional reduced Stein space, A a lower dimensional complex analytic subset with empty interior containing X_{sing}. Let Ω be an open relatively compact Stein domain in X and $K = \widehat{\overline{\Omega}_X}$ be the holomorphic convex hull of $\overline{\Omega}$ in X. Since X is Stein and $K = \widehat{K_X}$, K has a neighborhood basis of Oka-Weil domains in X (see Theorem 11, in [8], Volume III, page 102). Let X_0 be an Oka-Weil neighborhood of K in X, $X_0 \subset\subset X$. Then X_0 can be realized as a holomorphic subvariety of an open polydisk in some $\mathbb{C}^N$. Set $\Omega^* = \Omega \setminus A$. Let d_A be the distance to A, relative to an embedding of X_0 in $\mathbb{C}^N$ and let $|\ |$ and dV denote the induced norm on $\Lambda^{\cdot} T^*\Omega^*$, resp. the volume element (different embeddings of neighborhoods of $\overline{\Omega}$ in $\mathbb{C}^N$ give rise to equivalent distance functions and norms). For a measurable (p,q) form u on Ω^* set $\|u\|^2_{N,\Omega} := \int_{\Omega^*} |u|^2 d_A^{-N} dV$.

In this paper we address the question of whether we can solve the equation $\overline{\partial}u = f$ in Ω^* for a $\overline{\partial}$-closed (p,q) form f on Ω^* that vanishes to "high order" on A. Our main result is the following theorem:

Theorem 1.1. *Let X, Ω be as above. For every $N_0 \geq 0$, there exists $N \geq 0$ such that if f is a $\overline{\partial}-$closed $(p,q)-$form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is $v \in L^{2,\text{loc}}_{p,q-1}(\Omega^*)$ solving $\overline{\partial}v = f$, with $\|v\|_{N_0,\Omega'} < \infty$ for every $\Omega' \subset\subset \Omega$. For each $\Omega' \subset\subset \Omega$, there is a solution of this kind satisfying $\|v\|_{N_0,\Omega'} \leq C\|f\|_{N,\Omega}$, where C is a positive constant that depends only on Ω', N, N_0.*

When $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, Ω is Stein and $\overline{\Omega}$ has a Stein neighborhood, we obtain the following corollary of Theorem 1.1:

Corollary 1.2. *With N_0, N as in Theorem 1.1 and for f a $\overline{\partial}-$closed $(p,q)-$form on $\Omega^*, q > 0$, with $\|f\|_{N,\Omega} < \infty$, there is a solution u of $\overline{\partial}u = f$ on Ω^* with $\|u\|_{\Omega,N_0} \leq c\|f\|_{\Omega,N}$, c independent of f. In other words, we obtain a weighted L^2 estimate for u on all of Ω.*

Theorem 1.1 extends to the case when Ω is just holomorphically convex and contains a maximal compact subvariety B that is contained in A. It also extends to the case of (p,q) forms on X^* with values in a holomorphic vector bundle E over X. Theorem 1.1 and Corollary 1.2 can be used to construct analytic objects with prescribed behaviour on the

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maximal, positive dimensional compact subvariety B of a holomorphically convex manifold. We also expect them to be useful in studying the obstructions to solving $\overline{\partial}$ on a deleted neighborhood of an isolated singular point of a complex analytic set of dimension bigger than 2. The power series arguments that were used in the surface case in [3], [4] might be replaced by the solution of a Cousin problem with L^2 bounds which exists by our results when a finite number of obstructions vanish.

We have not managed to find a proof for Theorem 1.1 using transcendental L^2 methods. Instead, our arguments are based on resolution of singularities combined with cohomological arguments in the spirit of Grauert [5]. In particular, there exists a proper, holomorphic surjection $\pi : \widetilde{X} \to X$ with the following properties:

i) $\widetilde{X}$ is an $n-$dimensional complex manifold.
ii) $\tilde{A} = \pi^{-1}(A)$ is a hypersurface in $\tilde{\Omega}$ with only "normal crossing singularities", i.e. near each $x_0 \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \dots, z_n)$ in terms of which $\tilde{A}$ is given by $z_1 \cdots z_m = 0$, where $1 \leq m \leq n$
iii) $\pi : \widetilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism.

This follows from the following two facts: a) every reduced, complex space can be desingularized and, b) every reduced, closed complex subspace of a complex manifold admits an embedded desingularization. The exact statements and proofs can be found in [1], [2].

Let $\tilde{\Omega} := \pi^{-1}(\Omega)$. We give $\tilde{X}$ a Hermitian metric σ and we consider the corresponding distance function $d_{\tilde{A}}(x) = \mathrm{dist}(x, \tilde{A})$, volume element $d\tilde{V}_\sigma$ and norms on $\Lambda^{\cdot} T\tilde{X}$ and $\Lambda^{\cdot} T^*\tilde{X}$. Let J be the ideal sheaf of $\tilde{A}$ in $\tilde{X}$ and Ω^p the sheaf of holomorphic p forms on $\tilde{X}$. We shall consider the following sheaves on $\tilde{X}$ that are defined by:

$$\mathcal{L}_{p,q}(U) = \{u \in L^{2,\,\mathrm{loc}}_{p,\,q}(U);\ \ \overline{\partial} u \in L^{2,\,\mathrm{loc}}_{p,\,q+1}(U)\},$$

for every U open subset of $\tilde{X}$ and the obvious restriction maps $r^U_V : \mathcal{L}_{p,q}(U) \to \mathcal{L}_{p,q}(V)$, where $V \subset U$ are open subsets of $\tilde{X}$. Then $u \to \overline{\partial} u$ defines an $\mathcal{O}_{\tilde{X}}$-homomorphism $\overline{\partial}$: $\mathcal{L}_{p,q} \to \mathcal{L}_{p,q+1}$ and the sequence

$$0 \to \Omega^p \to \mathcal{L}_{p,0} \to \mathcal{L}_{p,1} \to \cdots \to \mathcal{L}_{p,n} \to 0$$

is exact by the local Poincaré lemma for $\overline{\partial}$. Since each $\mathcal{L}_{p,q}$ is closed under multiplication by smooth cut-off functions we have a fine resolution of Ω^p. In the same way, since J is locally generated by one function, the sequence

$$0 \to J^k\Omega^p \to J^k\mathcal{L}_{p,0} \to \cdots \to J^k\mathcal{L}_{p,n} \to 0$$

is a fine resolution of $J^k\Omega^p$. Here, $u \in (J^k\mathcal{L}_{p,q})_x$ if it can locally be written as $h^k u_0$ where h generates J_x and $u_0 \in (\mathcal{L}_{p,q})_x$. It follows that

$$H^q(\tilde{\Omega}, (J^k\Omega^p)_{|\Omega}) \cong \frac{\ker(\overline{\partial} : J^k\mathcal{L}_{p,q}(\tilde{\Omega}) \to J^k\mathcal{L}_{p,q+1}(\tilde{\Omega}))}{\overline{\partial}(J^k\mathcal{L}_{p,q-1}(\tilde{\Omega}))}. \tag{1}$$

Here is an outline of the proof of Theorem 1.1: The pullback $\pi^* f$ satisfies

$$\int_{\tilde{\Omega}} |\pi^* f|^2_\sigma d_{\tilde{A}}^{-N_1} d\tilde{V}_\sigma \leq C \int_{\Omega^*} |f|^2 d_A^{-N} dV, \tag{2}$$

for a suitable $0 < N_1 < N$ and $\overline{\partial}\pi^* f = 0$ on $\tilde{\Omega}$. Suppose for the moment that we could prove the following proposition:

Proposition 1.3. *For $q > 0$ and $k \geq 0$ given, there exists a natural number ℓ, $\ell \geq k$ such that the map*

$$i_* : H^q(\tilde{\Omega}, J^\ell \cdot \Omega^p) \to H^q(\tilde{\Omega}, J^k \cdot \Omega^p),$$

induced by the inclusion $i : J^\ell \cdot \Omega^p \to J^k \cdot \Omega^p$, is the zero map.

Using (2) we can show that $\pi^* f \in J^l \mathcal{L}_{p,q}(\tilde{\Omega})$ if $l \leq \frac{N_1}{2n}$. Assuming Proposition 1.3 this means that $\overline{\partial} v = \pi^* f$ has a solution in $J^k \mathcal{L}_{p,q-1}(\tilde{\Omega})$. Since $|h(x)| \leq C d_{\tilde{A}}(x)$ on compacts in the set where h generates J it follows that

$$\int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k}(x) d\tilde{V}_\sigma < \infty$$

where $\widetilde{\Omega}' = \pi^{-1}(\Omega')$ and $\Omega' \subset\subset \Omega$. Then $\overline{\partial}((\pi^{-1})^* v) = f$ on Ω^* and the final step will be to show that

$$\int_{\Omega'^*} |(\pi^{-1})^* v|^2 d_A^{-N_0} dV \leq c \int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k} d\tilde{V}_\sigma \tag{3}$$

when k is big enough.

Remark: Proposition 1.3 was inspired by Grauert [5] (Satz 1, Section 4). Grauert's result corresponds to the case where A is a finite set.

The paper is organized as follows: In section 2 we prove Proposition 1.3. Section 3, contains the estimates for the pullback of forms under π and π^{-1}. In Section 4 we prove Theorem 1.1. The proof of Corollary 1.2 is contained in section 5. Last but not least, in section 6 we discuss some generalizations to Theorem 1.1, Corollary 1.2.

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2. Proof of Proposition 1.3

Following Grauert [5], we consider more generally the coherent analytic sheaves $\mathcal{S}$ on $\tilde{X}$ that are torsion free i.e. sheaves with the property

$$T(\mathcal{S})_x = 0 \quad \text{for all } \; x \in \tilde{X} \tag{4}$$

where $T(\mathcal{S})_x = \{g_x \in \mathcal{S}_x : \; f_x \cdot g_x = 0 \text{ for some } f_x \neq 0, \; f_x \in \mathcal{O}_x \}$.

We shall show (Lemma 2.1) that when $\mathcal{S}$ is coherent and torsion free and $i : J^t\mathcal{S} \to \mathcal{S}$ is the inclusion homomorphism, then the induced map $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^t\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is zero when $q > 0$ and t is big enough. In order to exploit the idea that analytic sheaf cohomology on $\tilde{\Omega}$ is concentrated over $\tilde{A}$, the exceptional set of the resolution, we need to introduce the higher direct image sheaves, denoted by $R^q\pi_*\mathcal{S}$, of an analytic sheaf $\mathcal{S}$ on $\tilde{X}$, $q \geq 0$

and recall some basic facts about them. For $q \geq 0$ and $\mathcal{S}$ an $\mathcal{O}_{\tilde{X}}$-module, the higher direct image sheaves of $\mathcal{S}$ are the sheaves on X, associated to the presheaf

$$P : U \to H^q(\pi^{-1}(U), \mathcal{S})$$

where U open in X. When $\phi : \mathcal{S} \to \mathcal{S}'$ is an $\mathcal{O}_{\tilde{X}}$- homomorphism the induced maps $\phi_* : H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S}')$, U open in X, determine a sheaf homomorphism $\phi_\# : R^q\pi_*\mathcal{S} \to R^q\pi_*\mathcal{S}'$ on X. For future reference, we recall the $\mathcal{O}_X$-module structure on $R^q\pi_*\mathcal{S}$. Given U an open subset of X, $f \in \mathcal{O}_X(U)$, we define a map $f_U \bullet : \mathcal{S}_{|_{\pi^{-1}(U)}} \to \mathcal{S}_{|_{\pi^{-1}(U)}}$ described by $(f_U \bullet)s_x = (f \circ \pi)_x \cdot s_x$, $x \in \pi^{-1}(U)$, $s_x \in \mathcal{S}_x$ and let and $(f_U \bullet)_* : H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S})$ be the induced map on cohomology. We can then define a map $\mathcal{O}_X(U) \times H^q(\pi^{-1}(U), \mathcal{S}) \to H^q(\pi^{-1}(U), \mathcal{S})$ that sends $(f, c) \in \mathcal{O}_X(U) \times H^q(\pi^{-1}(U), \mathcal{S})$ to $(f_U \bullet)_* c$. It is easy to check that it is a morphism of presheaves $\mathcal{O}_X(-) \times H^q(\pi^{-1}(-), \mathcal{S}) \to H^q(\pi^{-1}(-), \mathcal{S})$ which extends naturally to a morphism on the associated sheaves $\mathcal{O}_X \times R^q\pi_*\mathcal{S} \to R^q\pi_*\mathcal{S}$.

The main theorem in Grauert [6], says that the direct image sheaves $R^q\pi_*\mathcal{S}$ are coherent $\mathcal{O}_X$-modules, when $\mathcal{S}$ is a coherent $\mathcal{O}_{\tilde{X}}$-module and $q \geq 0$. Since Ω is a Stein domain, Satz 5, Section 2 in [6], gives that the natural map $\pi_q : H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \to \Gamma(\Omega, R^q\pi_*\mathcal{S}_{|\Omega})$ is an isomorphism. This fact and the following lemma will enable us to finish the proof of Proposition 1.3.

Lemma 2.1. *For each $q > 0$ and for each coherent, torsion free $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$ there exists a $t \in \mathbb{N}$ such that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^t\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is the zero map, where $i : J^t\mathcal{S} \hookrightarrow \mathcal{S}$ is the inclusion map.*

Proof. We shall prove the lemma using downward induction on $q > 0$. Observe that $\tilde{\Omega}$ is an n-dimensional complex manifold with no compact n-dimensional connected components since it is obtained by blow-ups from a pure n-dimensional Stein space Ω. It follows from the Main Theorem in Siu [12] that $H^n(\tilde{\Omega}, \mathcal{S}) = 0$ for every coherent $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$. Hence, the statement is true for $q = n$ and any $t \in \mathbb{N}$.

When $q > 0$, Supp$R^q\pi_*\mathcal{S}$ is contained in A. The annihilator ideal $\mathcal{A}'$ of $R^q\pi_*\mathcal{S}$ is coherent and by Cartan's Theorem A there exist functions $f_1, \cdots, f_M \in \mathcal{A}'(X)$ that generate each stalk $\mathcal{A}'_z$ in a neighborhood of $\overline{\Omega}$. Let $\mathcal{A}$ be the $\mathcal{O}_{\tilde{X}}$-ideal generated by $\tilde{f}_j = f_j \circ \pi$, $1 \leq j \leq M$. A crucial observation which will be useful later, is that $(\tilde{f}_j)_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \to H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}})$ are zero for all j, $1 \leq j \leq M$, $q > 0$. To see this, consider the following commutative diagram

$$\begin{array}{ccc} H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) & \xrightarrow{(\tilde{f}_j)_{\tilde{\Omega},*}} & H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \\ \cong\big\downarrow & & \cong\big\downarrow \\ R^q\pi_*\mathcal{S}(\Omega) & \xrightarrow{(f_j)_{\Omega,\#}} & R^q\pi_*\mathcal{S}(\Omega) \end{array}$$

The vertical maps are isomorphisms, due to Satz 5, Section 2, in [6]. Recalling the way $\mathcal{O}_X$ acts on $R^q\pi_*\mathcal{S}$ and using the fact that the f_j's are in the annihilator ideal of $R^q\pi_*\mathcal{S}$ we conclude that $(f_j)_{\Omega,\#} = 0$. Hence, due to the commutativity of the above diagram $(\tilde{f}_j)_{\tilde{\Omega},*}$ is zero.

Let $Z(\mathcal{A})$ (resp. $Z(\mathcal{A}')$) denote the zero variety of $\mathcal{A}$ (resp. $\mathcal{A}'$). Since $Z(\mathcal{A}') = \mathrm{Supp} R^q\pi_*\mathcal{S}$ is contained in A, we have that $Z(\mathcal{A})$ is contained in $\tilde{A}$ near $\overline{\tilde{\Omega}}$. Thus by Rückert's Nullstellensatz for ideal sheaves, (see Theorem, page 82 in [7]), we have $J^\mu \subset \mathcal{A}$ on $\tilde{\Omega}$ for some $\mu \in \mathbb{N}$. Consider the surjection $\phi : \mathcal{S}^M \to \mathcal{A}\cdot\mathcal{S}$ given by $(s_1, \cdots, s_M) \to \sum_1^M \tilde{f}_j s_j$ and set $K = \ker\phi$. Clearly, K is torsion free, whenever $\mathcal{S}$ is. By definition the sequence

$$0 \to K \xrightarrow{i} \mathcal{S}^M \xrightarrow{\phi} \mathcal{A}\cdot\mathcal{S} \to 0 \tag{5}$$

is exact, and it follows from (4) and the fact that J is locally generated by one element that

$$0 \to J^a\cdot K \xrightarrow{i} J^a\cdot\mathcal{S}^M \xrightarrow{\phi} J^a\cdot\mathcal{A}\cdot\mathcal{S} \to 0$$

is also exact for any $a \in \mathbb{N}$.

Taking all the above into consideration we obtain the following commutative diagram:

$$\begin{tikzcd}
 & H^q(\tilde{\Omega}, J^{a+\mu}\cdot\mathcal{S}) \arrow[d] & \\
 & H^q(\tilde{\Omega}, J^a\cdot\mathcal{A}\cdot\mathcal{S}) \arrow[r, "\delta"] \arrow[d] & H^{q+1}(\tilde{\Omega}, J^a\cdot K) \arrow[d, "i_1"'] \\
H^q(\tilde{\Omega},\mathcal{S})^M \arrow[r, "\phi_{\tilde{\Omega},*}"] \arrow[dr, "\chi"] & H^q(\tilde{\Omega},\mathcal{A}\cdot\mathcal{S}) \arrow[r, "\delta"] \arrow[d, "i_2"] & H^{q+1}(\tilde{\Omega}, K) \\
 & H^q(\tilde{\Omega},\mathcal{S}) &
\end{tikzcd}$$

where the third row is exact (as part of the long exact cohomology sequence that arises from (5)) and the vertical maps are induced by sheaf inclusions. The map χ is defined to be $\chi := i_2 \circ \phi_{\tilde{\Omega},*}$ and we can show that $\chi(c_1, \cdots, c_M) = \sum_{j=1}^M (\tilde{f}_j)_{\tilde{\Omega},*} c_j$, where $c_j \in H^q(\tilde{\Omega},\mathcal{S})$, $1 \le j \le M$. It follows from the induction hypothesis for $(q+1)$ applied to the coherent, torsion-free sheaf K that there exists an integer a large enough such that $i_1 = 0$. Then, for an element $\sigma \in H^q(\tilde{\Omega}, \mathcal{A}\cdot\mathcal{S})$ that comes from $H^q(\tilde{\Omega}, J^{a+\mu}\cdot\mathcal{S})$, we have $\delta\sigma = 0$, so $\sigma = \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_M)$, $\sigma_j \in H^q(\tilde{\Omega},\mathcal{S})$, $1 \le j \le M$. By the crucial observation above and the way χ is defined, we conclude that χ is the zero map. Hence $i_2(\sigma) = i_2 \circ \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_m) = \sum_{j=1}^M (\tilde{f}_j)_{\tilde{\Omega},*}\sigma_j = 0$. Thus, for $i : J^{a+\mu}\cdot\mathcal{S} \hookrightarrow \mathcal{S}$ the inclusion map, we have that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^{a+\mu}\cdot\mathcal{S}) \to H^q(\tilde{\Omega},\mathcal{S})$ is the zero map. $\square$

Choosing as $\mathcal{S} := J^k\Omega^p$ we obtain Proposition 1.3.

3. Pointwise estimates for the pull back of forms under π, π^{-1}

Let σ be a metric on $\tilde{X}$, $|\ |_{x,\sigma}$ denote the pointwise norm of an element of $\wedge^r T_x\tilde{X}$ or $\wedge^r T_x^*\tilde{X}$ for some $r > 0$ with respect to the metric σ and $d_{\tilde{A}}$ the distance to $\tilde{A}$ in $\tilde{X}$. Let d_A denote the distance to A relative to an embedding of a neighborhood X_0 of $\overline{\Omega}$ in $\mathbb{C}^N$ and let $|\ |_y$ denote the pointwise norm of an element in $\wedge^r T_y(X_0 \setminus X_{\mathrm{sing}})$ for some $r > 0$, with respect to the

restriction of the pull back of the euclidean metric in $\mathbb{C}^N$ to $X_0 \setminus X_{\text{sing}}$. Let $dV,\ d\tilde{V}_\sigma$ denote the volume forms on $X_0 \setminus X_{\text{sing}}$, and $\tilde{X}$. The map $\pi : \tilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism of complex manifolds. It induces a linear isomorphism $\pi_* : \wedge^r T_x(\tilde{X} \setminus \tilde{A}) \to \wedge^r T_{\pi(x)}(X \setminus A)$ for $x \notin \tilde{A}$.

Lemma 3.1. *We have for $x \in \widetilde{\Omega} \setminus \tilde{A},\ v \in \wedge^r T_x(\widetilde{\Omega})$*

$$
\begin{aligned}
(6) \qquad & c'\, d^t_{\tilde{A}}(x) \ \leq\ d_A(\pi(x)) \leq C'\, d_{\tilde{A}}(x), \\
(7) \qquad & c\, d^M_{\tilde{A}}(x)\, |v|_{x,\sigma} \ \leq\ |\pi_*(v)|_{\pi(x)} \leq C\, |v|_{x,\sigma}.
\end{aligned}
$$

for some positive constants $c', c, C', C, t, M,$ where c, C, M may depend on r.

For an r-form a in Ω^* set $|\pi^* a|_{x,\sigma} := max\{\ | < a_{\pi(x)}, \pi_* v > |\ ;\ |v|_{x,\sigma} \leq 1,\ v \in \wedge^r T_x(\tilde{\Omega}\setminus\tilde{A})\}$, where by $<,>$ we denote the pairing of an r-form with a corresponding tangent vector. Using (7) we obtain:

$$
(8) \qquad c\, d^M_{\tilde{A}}(x)\, |a|_{\pi(x)} \leq |\pi^* a|_{x,\sigma} \leq C\, |a|_{\pi(x)}
$$

on $\tilde{\Omega}$, for some positive constant M.

Proof. The right hand side inequalities in the above estimates are obvious consequences of the differentiability of π, while the left hand side inequalities are consequences of the Lojasiewicz inequalities (see for example [10], or [11] Chapter 4, Theorem 4.1) in the following form:

Lemma 3.2. *Let S be a real analytic subvariety of some open subset V of $\mathbb{R}^d$ and let f be a real analytic, real-valued function in V. Let $Z_f = \{x \in V;\ f(x) = 0\}$. Then, for every compact $K \subset S$, there exist positive constants c, m such that*

$$
|f(x)| \geq c\, d(x, Z_f)^m
$$

when $x \in K$.

Lemma 3.2 generalizes easily to the case when S lies in a real analytic manifold and the distance is defined by a Riemannian metric.

To prove the left hand side inequality in (6) let $f : \tilde{X} \times A \to \mathbb{R}$ be given by $f(x, z) = |\pi(x) - z|^2$ and $K := \overline{\widetilde{\Omega}} \times$ (compact neighborhood of $\overline{\Omega} \cap A$). Clearly $Z_f \subset \tilde{A} \times A$. When $x \in \overline{\widetilde{\Omega}}$ and z is the nearest point to $\pi(x)$ in A, we have:

$$
f(x, z) = |\pi(x) - z|^2 = d(\pi(x), A)^2 \geq c\, d((x, z), Z_f)^m \geq c\, d_{\tilde{A}}(x)^m.
$$

If we write $m = 2t$ for some $t > 0$ constant, then we obtain from this last estimate the left hand side inequality in (6).

To prove the left hand side inequality in (7), we consider the unit sphere bundle $S^r(\tilde{X})$ in $\wedge^r T\tilde{X}$. We give $\tilde{X}$ a real analytic metric such that $S^r(\tilde{X})$ becomes a real analytic manifold. We choose a metric on $S^r(\tilde{X})$ such that the projection $p : S^r(\tilde{X}) \to \tilde{X}$ is distance

decreasing. For $\nu = (x, \xi_x)$ on the unit sphere bundle $S^r(\tilde{X})$, we set $f(\nu) := |\pi_*\xi_x|^2_{\pi(p(\nu))}$ and let $K := p^{-1}(\overline{\tilde{\Omega}})$. Clearly, $Z_f \subset p^{-1}(\tilde{A})$. It follows that $|\pi_*\xi_x|^2_{\pi(p(\nu))} = f(\nu) \geq c\, d(\nu, Z_f)^L \geq c\, d(p(\nu), \tilde{A})^L$ when $\nu \in K$. Write $2M = L$ for some $M > 0$ constant. The general case follows by applying this last inequality to $\frac{v}{|v|_x}$ for $v \neq 0$, $v \in \wedge^r T_x(\widetilde{\Omega})$.

Estimate (8) will be derived from (7) and the following remark:

Remark: Let $T : V \to W$ be a linear isomorphism of normed spaces such that $\|Tv\| \geq c\|v\|$ for $v \in V$ and $c > 0$ constant. Then $B_W(0, c) \subset T(B_V(0, 1))$, where by $B_V(0, 1)$ we denote the unit ball in V and $B_W(0, c)$ is the ball in W, centered at 0 and having radius c.

Using (7) and applying the above remark to $\pi_* : \wedge^r T_x(\tilde{\Omega} \setminus \tilde{A}) \to \wedge^r T_{\pi(x)}(\Omega \setminus A)$ we obtain for $x \in \tilde{\Omega} \setminus \tilde{A}$, $a_{\pi(x)} \in \wedge^r T^*_{\pi(x)}(\Omega \setminus A)$:

$$\begin{aligned} |\pi^* a|_{x,\sigma} &= \max\{\, | < a_{\pi(x)}, \pi_* v > | \ ; \ |v|_{x,\sigma} \leq 1, v \in \wedge^r T_x(\tilde{\Omega} \setminus \tilde{A})\} \\ &\geq \max\{\, | < a_{\pi(x)}, w > | \ ; \ |w|_{\pi(x)} \leq c\, d_{\tilde{A}}(x)^M,\ w \in \wedge^r T_{\pi(x)}(\Omega \setminus A)\} \\ &= c\, d_{\tilde{A}}(x)^M |a|_{\pi(x)}. \end{aligned}$$

This result applies in particular to the volume form in $\Omega \setminus A$ and gives:

$$c_1\, d_{\tilde{A}}(x)^{M_1} d\tilde{V}_{x,\sigma} \leq (\pi^* dV)_x \leq C_1\, d\tilde{V}_{x,\sigma}. \tag{9}$$

4. Proof of Theorem 1.1

Given $N_0 \in \mathbb{N}$, choose $k \geq M + t\,\frac{N_0}{2} \geq 0$, with t, M as in Lemma 3.1. Then by Proposition 1.3, there exists $\ell \geq k$ such that $H^q(\tilde{\Omega}, J^\ell\Omega^p) \to H^q(\tilde{\Omega}, J^k\Omega^p)$ is the zero homomorphism. Choose $N \in \mathbb{N}$ such that $N \geq 2n\ell + M_1$, where M_1 is as in (9).

The proof of theorem 1.1 will be based on the following change of variables result:

Lemma 4.1. *Let M, M' be orientable, Riemannian manifolds and $F : M \to M'$ an orientation preserving diffeomorphism. Let dV, dV' denote the corresponding volume elements of M, M' respectively. For $f \in L^1(M', dV')$ we have:*

$$\int_{M'} f dV' = \int_M (f \circ F)\ F^*(dV'). \tag{10}$$

Since $\pi : \tilde{\Omega} \setminus \tilde{A} \to \Omega \setminus A$ is a biholomorphism & orientation-preserving map-as long as we choose appropriate orientations on $\Omega \setminus A$, $\tilde{\Omega} \setminus \tilde{A}$-, for any f satisfying $\|f\|_{N,\Omega^*} < \infty$ we have (by applying Lemma 4.1):

$$\int_{\Omega\setminus A} |f|^2\, d_A^{-N}\, dV = \int_{\tilde{\Omega}\setminus\tilde{A}} |f|^2_{\pi(x)}\, d_A(\pi(x))^{-N} (\pi^* dV)_x.$$

Using the fact that

$$
\begin{aligned}
|f|_{\pi(x)} &\geq C^{-1}|\pi^* f|_{x,\sigma} && \text{(right hand side of (8))},\\
d_A(\pi(x))^{-1} &\geq C'^{-1} d_{\tilde{A}}^{-1}(x) && \text{(right hand side of (6))},\\
(\pi^* dV)_{x,\sigma} &\geq c_1 d_{\tilde{A}}^{M_1}(x)\, d\tilde{V}_{x,\sigma} && \text{(left hand side of (9))},
\end{aligned}
$$

we obtain

$$\|f\|^2_{N,\Omega^*} \geq c'' \int_{\tilde{\Omega}\setminus\tilde{A}} |\pi^* f|^2_{x,\sigma}\, d_{\tilde{A}}^{M_1-N} d\tilde{V}_{x,\sigma}$$

for some $c'' > 0$ constant. Since N was chosen such that $N \geq M_1$, we see that $\overline{\partial}\pi^* f = 0$ on $\tilde{\Omega}$. It is not hard to show that $\pi^* f \in J^\ell \mathcal{L}_{p,q}(\tilde{\Omega})$. By proposition 1.3 we know that there exists $v \in J^k \mathcal{L}_{p,q-1}(\tilde{\Omega})$ such that $\overline{\partial} v = \pi^* f$ in $\tilde{\Omega}$. Set $u := (\pi^{-1})^* v$. Then $\overline{\partial} u = f$ in Ω^* and for any $\Omega' \subset\subset \Omega$ we have:

$$
\begin{aligned}
\int_{\Omega'} |u|^2 d_A^{-N_0} dV &= \int_{\tilde{\Omega}'\setminus\tilde{A}} |u|^2_{\pi(x)}\, d_A^{-N_0}(\pi(x))\, \pi^*(dV)\\
&\lesssim \int_{\tilde{\Omega}'\setminus\tilde{A}} |v|^2_{x,\sigma}\, d_{\tilde{A}}^{-tN_0-2M}\, d\tilde{V}_{x,\sigma}\\
&\lesssim \int_{\tilde{\Omega}'\setminus\tilde{A}} |v|^2_{x,\sigma}\, d_{\tilde{A}}^{-2k}\, d\tilde{V}_{x,\sigma} < \infty.
\end{aligned}
$$

To pass from the 1st line to the 2nd one we use the fact that $|u|_{\pi(x)} \leq c^{-1} d_{\tilde{A}}^{-M}(x)\, |v|_{x,\sigma}$ $d_A^{-N_0}(\pi(x)) \leq c'^{-N_0} d_{\tilde{A}}^{-tN_0}(x)$ and that $(\pi^* dV)_{x,\sigma} \leq C_1\, d\tilde{V}_{x,\sigma}$.

To conclude the proof of Theorem 1.1 we shall need the following lemma:

Lemma 4.2. *Let M be a complex manifold and let E and F be Frechet spaces of differential forms (or currents) of type $(p, q-1)$, (p, q), whose topologies are finer than the weak topology of currents. Assume that for every $f \in F$, the equation $\overline{\partial} u = f$ has a solution $u \in E$. Then, for every continuous seminorm p on E, there is a continuous seminorm q on F such that the equation $\overline{\partial} u = f$ has a solution with $p(u) \leq q(f)$ for every $f \in F$, $q(f) > 0$.*

Proof. Set $G = \{(u, f) \in E \times F : \overline{\partial} u = f\}$. Then G is closed in $E \times F$. To see this, let $(u_\nu, f_\nu) \in G$ with $u_\nu \to u$ in E, $f_\nu \to f$ in F. For test forms $\phi \in C^\infty_{0,(n-p,n-q)}(X)$ we get

$$
\begin{aligned}
\int_M f \wedge \phi &= \lim_{\nu\to\infty} \int_M f_\nu \wedge \phi\\
&= \lim_{\nu\to\infty} (-1)^{p+q} \int_M u_\nu \wedge \overline{\partial}\phi = (-1)^{p+q} \int_M u \wedge \overline{\partial}\phi
\end{aligned}
$$

so $\overline{\partial} u = f$ weakly.

Thus, G is a Frechet space and the bounded surjection $\pi_2 : G \to F; (u, f) \to f$ must be open. The set $\pi_2(\{(u, v) \in G : p(u) < 1\})$ is an open neighborhood of 0 in F, and contains $\{f : q(f) \leq 1\}$ for some continuous seminorm q. Let $f \in F$, $0 < q(f) = c$.

Then $q(c^{-1}f) = 1$, so by the previous argument there exists a solution $c^{-1}u$ satisfying $\overline{\partial}(c^{-1}u) = c^{-1}f$ with $p(c^{-1}u) < 1$, i.e. $p(u) < c = q(f)$. □

When F is a Banach space with norm $\|.\|$, we conclude that, given a seminorm p, there is a constant $C > 0$ such that $\{f : \|f\| \le C^{-1}\} \subset \overline{\partial}(\{u : p(u) \le 1\}$, so $\overline{\partial}u = f$ has a solution u with $p(u) \le C\|f\|$. Applying this result to our situation, we see that if $\overline{\partial}f = 0$, $\|f\|_{\Omega,N} < \infty$ and $\Omega_0 \subset\subset \Omega$, we have a solution u of $\overline{\partial}u = f$ in $L^{2,\mathrm{loc}}_{p,q-1}(\Omega^*)$ with $\|u\|_{\Omega_0,N_0} \le c\|f\|_{\Omega,N}$.

5. Applications of Theorem 1.1

We apply Theorem 1.1 to the case where $A \cap \overline{\Omega}$ is a finite subset of $\overline{\Omega}$ with $b\Omega \cap A = \emptyset$, $\Omega \subset\subset X$ is Stein and $\overline{\Omega}$ has a Stein neighborhood Ω'.

Proposition 5.1. *With N_0, N as in Theorem 1.1 and $\overline{\partial}f = 0$ on Ω^* and $\|f\|_{\Omega,N} < \infty$, there is a solution u of $\overline{\partial}u = f$ on Ω^* with $\|u\|_{\Omega,N_0} \le c\|f\|_{\Omega,N}$, c independent of f. In other words, we obtain a weighted L^2 estimate for u on all of Ω.*

Proof. Choosing $\Omega_0 \subset\subset \Omega$ containing $A \cap \Omega$, we have a solution u_0 in $L^{2,\mathrm{loc}}_{p,q-1}(\Omega^*)$ with $\|u_0\|_{\Omega_0,N_0} \le c\|f\|_{\Omega,N}$. We introduce a cut-off function $\chi \in C^\infty(X)$ such that $\chi = 1$ on $X \setminus \Omega_0$ but $\chi = 0$ near $A \cap \Omega$. Set $f_1 = \overline{\partial}(\chi u_0)$. Clearly, $\|f_1\|_{L^2(\Omega)} \le c\|f\|_{\Omega,N}$ and $f_1 = 0$ near $\Omega \cap A$.

Let $\pi : \tilde{X} \to X$ be a desingularization of X and consider the equation $\overline{\partial}v = \pi^* f_1$ on $\tilde{\Omega}$. Let $\tilde{\Omega}_0 := \pi^{-1}(\Omega_0)$. The equation $\overline{\partial}v = \pi^* f_1$ is solvable in $L^2_{p,q-1}(\tilde{\Omega}_0)$. We can assume that $\tilde{\Omega}$ can be exhausted by smoothly bounded strongly pseudoconvex domains $\tilde{\Omega}_j := \{z \in \tilde{\Omega};\ \phi < c_j\ \}$ where c_j are real numbers, ϕ is an exhaustion function for $\tilde{\Omega}$, of class $C^3(\tilde{\Omega})$, strictly plurisubharmonic outside a compact subset and also that $b\tilde{\Omega}_0$ is smooth and strongly pseudoconvex and contained in each $\tilde{\Omega}_j$. To each $\tilde{\Omega}_j$ we apply Theorem 3.4.6 in [9] and we obtain a solution v_j to the equation $\overline{\partial}v_j = \pi^* f_1$ in $\tilde{\Omega}_j$ with

$$\int_{\tilde{\Omega}_j} |v_j|^2\, e^{-\phi}\, d\tilde{V}_\sigma \le C \int_{\tilde{\Omega}} |\pi^* f_1|^2\, d\tilde{V}_\sigma$$

where C is a positive constant independent of j, f (this follows from a careful inspection of the proof of Theorem 3.4.6 in [9]).

Consider the trivial extensions v_j^o of v_j outside $\tilde{\Omega}_j$. Let v be a weak limit of v_j^o. Then

$$\int_{\tilde{\Omega}} |v|^2\, e^{-\phi}\, d\tilde{V}_\sigma \le C \int_{\tilde{\Omega}} |\pi^* f_1|^2\, d\tilde{V}_\sigma$$

and $\overline{\partial}v = \pi^* f_1$ in $\tilde{\Omega}$. So there is a solution v satisfying $\|v\|_{L^2(\tilde{\Omega})} \le c\|f_1\|$. Then $w := (\pi^{-1})^* v$ satisfies $\overline{\partial}w = f_1$ in Ω^* but we have no longer control of its L^2-norm near $A \cap \Omega$. Choose another cut-off function χ_0 such that $\chi_0 = 1$ on suppχ but $\chi_0 = 0$ near $\Omega \cap A$. Then

$$\begin{aligned}\overline{\partial}((1-\chi)u_0 &+ \chi_0(\pi^{-1})^*v) = \\ &= (1-\chi)f - \overline{\partial}\chi \wedge u + \overline{\partial}\chi_0 \wedge (\pi^{-1})^*v + \\ &+ \chi f + \overline{\partial}\chi \wedge u = \\ &= f + \overline{\partial}\chi_0 \wedge (\pi^{-1})^*v\end{aligned}$$

Finally we may solve $\overline{\partial}v_1 = \overline{\partial}\chi_0 \wedge (\pi^{-1})^*v$ in Ω'^* (apply Theorem 1.1 to the trivial extension of $\overline{\partial}\chi_0 \wedge (\pi^{-1})^*v$ in Ω'):

$$\|v_1\|_{\Omega,N_0} \leq c\,\|\overline{\partial}\chi_0 \wedge (\pi^{-1})^*v\|_{\Omega',N} \leq c'\,\|\overline{\partial}\chi_0 \wedge (\pi^{-1})^*v\|_{L^2(\Omega)} \leq C\|f\|_{\Omega,N}$$

since $\overline{\partial}\chi = 0$ near A. Thus, $u := (1-\chi)u_0 + \chi_0\,(\pi^{-1})^*v - v_1$ is a solution with the required estimate. □

6. Generalizations

Theorem 1.1 and Corollary 1.2 also extend to the case when Ω is a relatively compact domain in a complex space X of pure dimension n with strictly pseudoconvex boundary. We know that Ω contains a maximal positive dimensional compact variety B and let A be a nowhere open analytic subvariety of X containing X_{sing} and B. Then theorem 1.1 carries over verbatim to the case described above. The proof needs the following modifications: Let $\overline{\Omega} \subset X_0$ be a neighborhood with strictly pseudoconvex boundary and maximal positive dimensional compact subvariety B. Take the Remmert reduction $\phi : X_0 \to X_1$ so that X_1 is Stein, $\phi(B) = B_1$ is finite and $\phi : X_0 \setminus B_0 \to X_1 \setminus B_1$ is a biholomorphism. Let $\pi : \tilde{X_0} \to X_0$ be a desingularization of X_0 such that $\pi^{-1}(A)$ is a hypersurface with normal crossings. To obtain a proof of Proposition 1.3 (vanishing cohomology), we need to consider direct images $R^q(\phi\circ\pi)_*\mathcal{S}$ on the Stein space X_1 and their annihilator ideal $\mathcal{A}$ for $\mathcal{S}$ coherent on $\tilde{X}$. Then, the proof carries over.

Corollary 1.2, for the case when $X_{\text{sing}} \cap b\Omega$ is empty, with $A = B \cup (X_{\text{sing}} \cap \Omega)$ follows exactly as above.

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